YoungDiagram

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Chapter 1

Genes and Chromosomes

1.1 Gene types and genes

Definition 1 (Gene type)**.** A *gene type* is one of three polarization types: `NonPolarized`, `Positive`, or `Negative`. There is a natural involution $\varepsilon \mapsto -\varepsilon$ that swaps `Positive` and `Negative` and fixes `NonPolarized`.

Definition 2 (Gene)**.** A *gene* $g = (n, \varepsilon)$ consists of a *rank* $n \in \mathbb{N}$ and a *type* $\varepsilon \in$ `GeneType`.

Definition 3 (Gene signature)**.** The *signature* of a gene $g = (n, \varepsilon)$ is a pair $(a, b) \in \mathbb{Q} \times \mathbb{Q}$ defined by:

- If $\varepsilon =$ `NonPolarized`: $(a, b) = (n/2,\, n/2)$.
- If $\varepsilon =$ `Positive`: $(a, b) = (\lceil n/2 \rceil,\, \lfloor n/2 \rfloor)$.
- If $\varepsilon =$ `Negative`: $(a, b) = (\lfloor n/2 \rfloor,\, \lceil n/2 \rceil)$.

Lemma 4 (Signature sum equals rank)**.** *For any gene g, we have $a(g) + b(g) = \mathrm{rank}(g)$.*

1.2 Chromosomes

Definition 5 (Chromosome)**.** A *chromosome* is a finitely supported function $X \colon$ `Gene` $\to \mathbb{N}$, i.e. a formal sum of genes with non-negative integer multiplicities. We write `Chromosome` $=$ `Gene` $\to_0 \mathbb{N}$.

Definition 6 (Chromosome signature)**.** The *signature* of a chromosome X is the additive extension: $\mathrm{sig}(X) = \sum_g X(g) \cdot \mathrm{sig}(g) \in \mathbb{Q} \times \mathbb{Q}$.

Definition 7 (Rank)**.** The *rank* of a chromosome X is $\mathrm{rank}(X) = \sum_g X(g) \cdot \mathrm{rank}(g)$.

Lemma 8 (Signature sum equals rank for chromosomes)**.** *For any chromosome X, the sum of the two signature components equals the rank:* $\mathrm{sig}(X).1 + \mathrm{sig}(X).2 = \mathrm{rank}(X)$.

Definition 9 (Dominance order)**.** For chromosomes X and Y, we say X *dominates* Y (written $Y \le X$) if $\mathrm{sig}(X^{(k)}) \le \mathrm{sig}(Y^{(k)})$ componentwise for all $k \ge 0$, where $X^{(k)}$ denotes the k-th derivative.

1.3 Prime and lift operations

Definition 10 (Prime (derivative))**.** The *prime* (derivative) operation $X \mapsto X'$ is the additive homomorphism that maps a gene (n, ε) with $n \geq 1$ to $(n-1, -\varepsilon)$, and sends genes of rank 0 to 0.

Definition 11 (Lift (antiderivative))**.** The *lift* operation is an additive homomorphism mapping a gene (n, ε) to $(n+1, -\varepsilon)$.

Lemma 12 (Prime-lift left inverse)**.** *The prime operation is a left inverse to lift:* $(\operatorname{lift} X)' = X$ *for all* X*.*

Lemma 13 (Max rank decreases under prime)**.** *If* $X \neq 0$*, then* $\operatorname{maxRank}(X') < \operatorname{maxRank}(X)$*.*

1.4 Decompositions

Definition 14 (Below and above)**.** For $k \in \mathbb{N}$, $X_{\leq k}$ (below) filters genes with rank $\leq k$, and $X_{>k}$ (above) filters genes with rank $> k$. We have $X = X_{\leq k} + X_{>k}$.

Definition 15 (Parity decomposition)**.** Every chromosome decomposes as $X = X_{\text{odd}} + X_{\text{even}}$ where X_{odd} (resp. X_{even}) collects genes of odd (resp. even) rank.

Lemma 16 (Parity and prime interchange)**.** $(X')_{\text{even}} = (X_{\text{odd}})'$ *and* $(X')_{\text{odd}} = (X_{\text{even}})'$*.*

Chapter 2

Varieties

2.1 Filtered varieties

Definition 17 (Variety)**.** A *variety* is an additive submonoid of `Chromosome`.

Definition 18 (Filtered variety)**.** Given a predicate p on genes, the *filtered variety* V_p consists of all chromosomes whose support satisfies p.

Lemma 19 (Filtered varieties are prime-stable)**.** *If p is lift-stable (i.e. $p(g)$ iff $p(\mathrm{lift}(g))$), then the prime of a filtered variety is itself: $V_p' = V_p$.*

2.2 The polarized variety Pi

Definition 20 (Polarized chromosome)**.** A chromosome X is *polarized* if every gene in its support has type $\neq$ `NonPolarized`.

Definition 21 (Variety Pi)**.** The variety Pi is the filtered variety of all polarized chromosomes.

Lemma 22 (Pi is prime-stable)**.** $\mathrm{Pi}' = \mathrm{Pi}$.

Lemma 23 (Polarized signature is natural)**.** *If $X \in \mathrm{Pi}$, the signature components of X are natural numbers.*

Lemma 24 (Sigma sequence determines element of Pi)**.** *If $X, Y \in \mathrm{Pi}$ satisfy $\mathrm{sig}(X^{(k)}) = \mathrm{sig}(Y^{(k)})$ for all $k \geq 0$, then $X = Y$.*

Lemma 25 (Antisymmetry of dominance on Pi)**.** *For $X, Y \in \mathrm{Pi}$, if $X \leq Y$ and $Y \leq X$, then $X = Y$. Hence Pi carries a partial order.*

2.3 The non-polarized variety Lambda

Definition 26 (Non-polarized chromosome)**.** A chromosome is *non-polarized* if every gene in its support has type `NonPolarized`.

Definition 27 (Variety Lambda)**.** The variety Lambda is the filtered variety of all non-polarized chromosomes.

Lemma 28 (Lambda is prime-stable)**.** $\mathrm{Lambda}' = \mathrm{Lambda}$.

2.4 Mixed varieties and labels

Definition 29 (Mixed variety)**.** Given a pair of varieties (V_1, V_2), the *mixed variety* $\mathrm{Mix}(V_1, V_2)$ consists of all chromosomes whose odd part lies in V_1 and even part lies in V_2.

Definition 30 (Five labeled varieties)**.** The five labeled varieties $V_0, \dots, V_4$ indexed by $\mathrm{Fin}\,5$ are:

- $V_0 = \mathrm{Pi}$
- $V_1 = \mathrm{Mix}(\mathrm{Pi}, \mathrm{Lambda})$
- $V_2 = \mathrm{Lambda}$
- $V_3 = \mathrm{Mix}(\mathrm{Lambda}, \mathrm{Pi})$
- $V_4 = \mathrm{Mix}(\mathrm{Lambda}, \mathrm{Lambda})$

Definition 31 (Label prime permutation)**.** The prime operation on varieties induces a permutation of the five labels: $V_i' = V_{\pi(i)}$.

Chapter 3

Sigma Sequences

Definition 32 (Sigma sequence)**.** For a chromosome X, the *sigma sequence* is the function $\sigma_X(k) = \text{sig}(X^{(k)}) \in \mathbb{Q} \times \mathbb{Q}$. We write $a_X(k) = \sigma_X(k).1$ and $b_X(k) = \sigma_X(k).2$.

Lemma 33 (Sigma is antitone)**.** *The sigma sequence is antitone (componentwise decreasing).*

Lemma 34 (Sigma is eventually zero)**.** *There exists K such that $\sigma_X(k) = 0$ for all $k \geq K$.*

Lemma 35 (Interlacing conditions)**.** *The components of the sigma sequence satisfy interlacing inequalities. If k is even, then $b_X(k+1) \leq a_X(k)$ and $a_X(k+1) \leq b_X(k)$. If k is odd, the inequalities are reversed.*

Lemma 36 (Difference conditions for Pi)**.** *If $X \in \text{Pi}$, then $a_X(k) - b_X(k+1) \leq 1$ and $b_X(k) - a_X(k+1) \leq 1$.*

Lemma 37 (Dominance implies componentwise inequality)**.** *If $X < Y$ in Pi, then $\sigma_X(k) \leq \sigma_Y(k)$ componentwise for all k with strict inequality for some k.*

Chapter 4

Mutations

4.1 Definition of mutations

Definition 38 (Mutation)**.** A pair (X, Y) of chromosomes forms a *mutation* if:

1. $X \neq Y$,
2. $\text{sig}(X) = \text{sig}(Y)$,
3. $Y \leq X$ (dominance).

4.2 Primitive mutations on Pi

Definition 39 (Primitive mutations of type 1)**.** Given ε, $k \leq m$: $X_1 = (m, \varepsilon) + (k, -\varepsilon)$ and $Y_1 = (m+1, -\varepsilon) + (k-1, \varepsilon)$ (when $k \geq 1$).

Definition 40 (Primitive mutations of type 2)**.** Given ε, $k \leq m$: $X_2 = (m, \varepsilon) + (k, \varepsilon)$ and $Y_2 = (m+1, \varepsilon) + (k-1, \varepsilon)$ (when $k \geq 1$).

Definition 41 (Primitive mutations of type 3)**.** Given ε, $k \leq m$: $X_3 = (m, \varepsilon) + (k, -\varepsilon)$ and $Y_3 = (m+2, -\varepsilon) + (k-2, \varepsilon)$ (when $k \geq 2$).

Definition 42 (Primitive mutation relation)**.** A *primitive mutation* on Pi is one of the three types above (up to adding a common chromosome Z).

Definition 43 (Step relation)**.** A *step* on Pi is a primitive mutation plus an arbitrary summand: $\text{Step}(X + Z, Y + Z)$ whenever $\text{Primitive}(X, Y)$.

4.3 Properties of mutations on Pi

Lemma 44 (Primitive mutations are mutations)**.** *Every primitive mutation satisfies the mutation conditions.*

Lemma 45 (Type 1 signature preservation)**.** *Type 1 mutations preserve the signature at all levels:* $\text{sig}(X_1^{(k)}) = \text{sig}(Y_1^{(k)})$ *for all* k.

Lemma 46 (Type 1 dominance)**.** *Type 1 mutations satisfy* $Y_1 \leq X_1$.

Lemma 47 (Type 2 signature preservation)**.** *Type 2 mutations preserve the signature at all levels.*

Lemma 48 (Type 2 dominance)**.** *Type 2 mutations satisfy* $Y_2 \leq X_2$.

Lemma 49 (Type 3 signature preservation)**.** *Type 3 mutations preserve the signature at all levels.*

Lemma 50 (Type 3 dominance)**.** *Type 3 mutations satisfy* $Y_3 \leq X_3$.

4.4 Main mutation theorem

Theorem 51 (Existence of mutations)**.** *For any* $X, Y \in \mathrm{Pi}$ *with* $\mathrm{rank}(X) = \mathrm{rank}(Y) = n$ *and* $Y \leq X$*, either* $X = Y$ *or there exists a step mutation from* X *towards* Y*: there exist* $X' \in \mathrm{Pi}$ *with* $\mathrm{Step}(X, X')$ *and* $Y \leq X' < X$.

Chapter 5

Lifting Theorem

Lemma 52 (Mutation lifting)**.** *Given a mutation* (X, Y) *in variety* V_i *with* $X' = Y'$*, there exists* $Z \in V_{\pi(i)}$ *such that* (Z, Y) *is a mutation in* $V_{\pi(i)}$ *and* $Z' = X$*.*